THE M-CENTER PROBLEM

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THESIS

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ABSTRACT

Solution algorithms are presented for the vertex m-center and the absolute m-center problem. Both algorithms use partitioning techniques. The algorithms use special properties of the max-min node to test for optimality. The vertex m-center algorithm establishes an order among all partitions of a graph according to the smallest vertex m-radius each partition can have. It then directs one to calculate the vertex m-radii only for those partitions which can provide a minimal vertex m-radius. The absolute m-center algorithm establishes an initial solution which may not be optimal. Other partitions are then tested against this solution to determine whether or not they provide a better solution. A point is reached at which no untested partition can improve the extant solution and the algorithm terminates.

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I. INTRODUCTION

This thesis is concerned with the m-center problem of graph theory. It is desired to find m points on a graph which are closest in some sense to the nodes of the graph. If the m points are constrained to be nodes they are referred to as the vertex m-centers of the graph. If the m points are allowed to be anywhere on the graph, they are called absolute m-centers. Reed [3] presents a comprehensive summary of previous work done to formulate and solve the problem in various contexts.

Analytical definitions for the vertex m-centers and absolute m-centers are given in section II, together with supporting definitions and notational conventions.

Section III presents a solution algorithm for the vertex m-center problem which can be used if m is greater than one. The algorithm is original in detail, to the best of the author's knowledge, although its general concept is not [1,2].

Section IV contains the author's main contribution. A series of theorems is presented which leads to an absolute m-center algorithm. The algorithm guarantees that an optimal solution will be found and works directly from the vertex m-center algorithm delineated in section III. It is also shown that the absolute m-center algorithm of Rosenthal and Smith [4] does not provide an optimal answer in general.

Section V summarizes the thesis and makes suggestions for further research.

II. DEFINITIONS AND NOTATIONS

Unless stated otherwise, the context of any definition is the connected graph $G(N,A)$, where N is a non-empty set of nodes (vertices) and A is the set of undirected arcs incident to the nodes in N. Let X be the set of all points on $G(N,A)$.

The two-tuple (i,j) denotes the shortest path in $G(N,A)$ between points i and j, where the length of a path is the sum of the magnitudes associated with the arcs comprising the path. Let $d(i,j)$ represent the length of (i,j). For any intermediate point k on the path between i and j,

$$(i,j) \equiv (i,k,j)$$

and

$$d(i,j) = d(i,k) + d(k,j).$$

DEFINITION 1:

The distance matrix of $G(N,A)$ is the square matrix $\left\| d_{ij} \right\|$,

where

$$d_{ij} = d(i,j).$$

Let S be the set of all spanning trees of $G(N,A)$. For S_k, an element of S, let $(i,j)_k$ be the shortest path between nodes i and j in the spanning tree S_k, and let $d(i,j)_k$ be its length. In general,

$$d(i,j) \leq d(i,j)_k. \qquad (1)$$

This is true since removing arcs from $G(N,A)$ can, at best, leave (i,j) unchanged.

DEFINITION 2:

a. The vertex center of a graph $G(N,A)$ is that vertex $C(v) \in N$, such that

$$\max_{i} d(i,C(v)) = \min_{j} \max_{i} d(i,j), \; i,j \in N.$$

b. The vertex radius of a graph $G(N,A)$ is $R(v)$, where

$$R(v) = \max_{i} d(i,C(v)), \; i \in N.$$

DEFINITION 3:

The central path of $G(N,A)$ is the path $(I,J)_K$, such that

$$d(I,J)_K = \min_{k} \max_{i,j} d(i,j)_k, \; i,j \in N, S_k \in S.$$

Rosenthal and Smith [4] were the originators of the central path concept. Their definition of the central path is subject to awkwardness caused by the possible existence of loops or circuits in a general graph. The author chose to define the central path in terms of spanning trees so as to avoid that difficulty.

The spanning tree from which the central path is derived need not be unique and has several important properties. Let $(i,j)^*$ denote the path between node i and node j in the central-path spanning tree, and let $d(i,j)^*$ be its length. Because of (1), it follows that

$$d(i,j) \leq d(i,j)^*. \tag{2}$$

DEFINITION 4:

a. The central-path vertex center of a graph $G(N,A)$ is that vertex $C(v,cp) \in N$, such that

$$\max_{i} d(i,C(v,cp))^{*} = \min_{j} \max_{i} d(i,j)^{*}, \; i,j \in N.$$

b. The central-path vertex radius of a graph $G(N,A)$ is $R(v,cp)$, where

$$R(v,cp) = \max_{i} d(i,C(vcp))^{*}, \; i \in N.$$

THEOREM 1:

For a graph $G(N,A)$,

$$R(v) \leq R(v,cp).$$

Proof: It has been shown that

$$d(i,j) \leq d(i,j)^{*},$$

therefore

$$\max_{i} d(i,j) \leq \max_{i} d(i,j)^{*}, \; i,j \quad N.$$

It then follows that

$$\min_{j} \max_{i} d(i,j) \leq \min_{j} \max_{i} d(i,j)^{*}, \; i,j \quad N.$$

therefore

$$R(v) \leq R(v,cp).$$

DEFINITION 5:

a. The absolute center of the graph $G(N,A)$ is that point $C(ab) \quad X$ such that

$$\max_{i} d(i,C(ab)) = \min_{x \in X} \max_{i} (i,x), \; i \in N.$$

b. The absolute radius of the graph G(N,A) is R(ab), where

$$R(ab) = \max_i d(i, C(ab)),\ i \in N.$$

THEOREM 2 [4]:

The absolute center of the graph G(N,A) must lie at the midpoint of the central path of G(N,A).

If C(ab,cp) denotes the central-path spanning-tree, absolute center of G(N,A) and R(ab,cp) represents the central-path spanning-tree, absolute radius, then Theorem 2 implies that

$$C(ab) \equiv C(ab,cp)$$

and

$$R(ab) \equiv R(ab,cp).$$

DEFINITION 6:

The max-min node is that node $\overline{V} \in N$ such that

$$(\overline{V}) = \min_{i \neq \overline{V}} d(i, \overline{V}) = \max_j \min_{i \neq j} d(i,j)$$

It will be necessary at times to deal explicitly with distances from various nodes to the max-min node. Since a graph can have several max-min nodes the following convention will be used. Let $\overline{V}(A,B/I)$ denote the distance from node A to node B, where node B is a max-min node of the graph and I is an ordering index. Then for any $\overline{V}(a,b/m)$ and $\overline{V}(c,d/n)$, if $m < n$, then

$$\overline{V}(a,b/m) \leq \overline{V}(d,d/n).$$

DEFINITION 7:

The graph $G(N,A)$ is said to be partitioned if it is broken into a number of subgraphs.

Let M be a set of m nodes chosen from $G(N,A)$. The set M will be called a partition of $G(N,A)$. For each $i \in N$, $i \notin M$, place i in the set N_j associated with $j \in M$ if

$$d(i,j) = \min_k d(i,k), \ k \in M,$$

thus establishing a subgraph $G(N_j,A)$ of $G(N,A)$ for node j. Each node $j \in M$ will be called a partition root of $G(N_j,A)$.

The vertex center and vertex radius of $G(N_j,A)$ will be $C_j(v)$ and $R_j(v)$ respectively where

$$R_j(v) = \max_{i \in N_j} d(i,C_j(v)) = \min_{k \in N_j} \max_{i \in N_j} d(i,k).$$

The absolute center and absolute radius of $G(N_j,A)$ will be $C_j(ab)$ and $R_j(ab)$ respectively where, for X_j the set of points on $G(N_j,A)$,

$$R_j(ab) = \max_{i \in N_j} d(i,C_j(ab)) = \min_{x \in X_j} \max_{i \in N_j} d(i,x).$$

Let R_j be the root length of $G(N_j,A)$, defined by

$$R_j = \max_i d(i,j), \ i \in N_j.$$

Let $\underline{R}_M$ be the partition radius of $G(N,A)$ where

$$\underline{R}_M = \max_j R_j, \ j \in M.$$

The absolute partition radius will be denoted $\underline{R}_M(ab)$ where

$$\underline{R}_M(ab) = \max_j R_j(ab), \quad j \in M. \tag{3}$$

DEFINITION 8:

The vertex m-center of the graph G(N,A) is that set of m nodes $M^* \subset N$, such that $\underline{R}_M$ is minimized. That is, if R(m,v) denotes the vertex m-center radius, then

$$R(m,v) = \min_{M \subset N} \underline{R}_M .$$

DEFINITION 9:

Let X_M be a set of m points on G(N,A) and define

$$d(i,X_M) = \min_x d(i,x), \quad i \in N, \ x \in X_M .$$

The absolute m-center of the graph G(N,A) is that set of m points $X_M^* \subset X$, such that

$$\max_{i \in N} d(i,X_M^*) = \min_{X_M \subset X} \max_{i \in N} d(i,X_M).$$

Let R(m,ab) denote the absolute m-center radius, then

$$R(m,ab) = \max_{i \in N} d(i,X_M^*).$$

Alternatively, the absolute m-center radius may be defined as

$$R(m,ab) = \min_{M \subset N} \underline{R}_M(ab).$$

Finally, let S_K be the spanning-tree from which the central-path of G(N,A) is chosen. Let C(ab) signify the absolute center of G(N,A). Given two nodes, i and j, j will be <u>interior</u> to i if and only if $(i,C(ab))_K$

is identically $(i, j, C(ab))_K$. Conversely, j will be <u>exterior</u> to i if and only if i is interior to j. The node i will be an <u>end-node</u> of a path in S_K if and only if it is not interior to any other node in S_K.

III. VERTEX M-CENTER ALGORITHM

Reed [3] develops what might be called the brute force approach to solving for the vertex m-center of a graph. Using this method it is necessary to calculate $\underline{R}_M$ for all possible partitions of G(N,A) and then to compare the $\underline{R}_M$ against each other to determine the minimum.

The algorithm to be presented in this section appears to be more efficient. It makes use of the max-min node to pick those partitions of G(N,A) which might produce the minimum $\underline{R}_M$. For any partition of G(N,A) which does not have $\overline{V}$ as a partition root, the minimum distance from any root of the partition to $\overline{V}$ represents a lower bound on $\underline{R}_M$. Therefore the algorithm first finds $\underline{R}_M$ for those partitions in which $\overline{V}$ is a root vertex. If it is then necessary to calculate $\underline{R}_M$ for any other partitions, the procedure tests partitions according to their increasing distance to $\overline{V}$. Any partition having its distance to $\overline{V}$ greater than the R(v) is automatically rejected as a possible solution.

A. THE ALGORITHM

For a given graph G(N,A) with distance matrix $\|d_{ij}\|$:

1. Find those nodes which quality as $\overline{V}$ and note the value $(\overline{V})$.

2. Determine $\underline{R}_M$ for every partition having $\overline{V}$ as a partition root. If more than one node qualifies as $\overline{V}$ this step must be performed for each qualifying vertex.

3. Find the smallest $\underline{R}_M$ among those calculated in step 2 and test it in the following manner:

a. If min $\underline{R}_M < (\overline{V})$, go to step 4.

b. If min $\underline{R}_M = (\overline{V})$, go to step 5.

c. If min $\underline{R}_M > (\overline{V})$, go to step 7.

4. Set R(m,v) equal to min $\underline{R}_M$. Any partition among those determined in step 3 having $\underline{R}_M$ = R(m,v) is a vertex m-center for the graph. Since any partition not identified in step 3 must have $\underline{R}_M$ no smaller than $(\overline{V})$, the algorithm terminates.

5. Set R(m,v) equal to min $\underline{R}_M$. Any partition among those determined in step 3 having $\underline{R}_M$ = R(m,v) is a vertex m-center for the graph. At this point a solution has been found and one or several partitions identified as vertex m-centers. Other partitions not having $\overline{V}$ as a root vertex may qualify as vertex m-centers also.

6. Find those nodes $j \in N$ such that

$$d(j,\overline{V}) = (\overline{V}).$$

Again, this must be done for all those nodes qualifying as $\overline{V}$. Determine $\underline{R}_M$ for all partitions having any such node j as a partition root. Any partition so determined whose $\underline{R}_M$ is equal to the value of R(m,v) from step 5 qualifies as a vertex m-center of the graph. Since all other partitions must have $\underline{R}_M$ greater than $(\overline{V})$, the algorithm terminates.

7. It is necessary at this point to order the nodes of G(N,A) in terms of their distance to $\overline{V}$. Suppose p and q were $\overline{V}$ nodes, then following the convention of definition 6, the list of values might be:

$\overline{V}(i,p/1)$, $\overline{V}(j,q/2)$, $\overline{V}(k,p/3)$, ...

The zero values should not be included in this ordering. It is possible for $\overline{V}(i,p/1)$ to equal $\overline{V}(j,q/2)$. If this should happen, the remaining steps of the algorithm should be performed for all nodes which have an identical distance to $\overline{V}$.

8. Start with the minimal $\overline{V}(A,B/I)$. Calculate $\underline{R}_M$ for all those partitions having node A as a root vertex such that

$$d(A,B) = \min_{v \in M} d(v,B).$$

If the minimum of the $\underline{R}_M$ so calculated is greater than the $\overline{V}(A,B/I)$ used to begin this step, choose the next largest $\overline{V}(A,B/I)$ and calculate the appropriate $\underline{R}_M$. Continue doing this until

$$\min \underline{R}_M = \overline{V}(A,B/I).$$

9. Set $R(m,v) = \min \underline{R}_M$ as determined in step 8. Any partition for which $\underline{R}_M$ was calculated during any of the algorithm steps such that

$$\underline{R}_M = R(m,v)$$

qualifies as a vertex m-center of the graph. Since any partition not considered already must have $\underline{R}_M$ greater than the value $\overline{V}(A,B/I)$ of step 8, the solution terminates.

B. FIRST EXAMPLE

Consider the graph in Figure 1. It is desired to find the vertex two-centers. Step 1 is to find $\overline{V}$ and $(\overline{V})$. Figure 2 shows the distance matrix for this graph. Each element in the row labeled "min," directly under the matrix, is the

$$\min_{i \neq j} d(i,j)$$

for the jth column. The underlined element in the min row is the max of the minimum values and identifies vertex 2 as $\overline{V}$. The value $(\overline{V})$ is seven.

Figure 3 shows the partition distance matrix for those partitions having vertex 2 as a partition root. The column to the right of the matrix, labeled $\underline{R}_M$, has elements which are the maximum values from each corresponding row of the matrix.

Since

$$\min \underline{R}_M = 7 = (\overline{V}),$$

the algorithm branches to step 5 resulting in $R(2,v) = 7$ and the partition $\{2,5\}$ being a vertex two-center for this graph.

The algorithm cautions that other vertex two-centers may exist. This may or may not be important depending on the context of the problem. Looking at the distance matrix in Figure 2 it is seen that vertices 1 and 5 have distances to $\overline{V}$ equal to seven. Proceeding with step 6, Figure 4 shows the partition distance matrix for those partitions having nodes 1 and 5 as roots. From the $\underline{R}_M$ column it is seen that five other vertex two-centers exist for this graph and the algorithm terminates.

C. SECOND EXAMPLE

The graph for this example is shown in Figure 5, together with its associated distance matrix. The value corresponding to $(\overline{V})$ is

underlined as before. Figure 6 shows the partition distance matrix for those partitions having node 5 as a root. It is seen that min $\underline{R}_M = (\overline{V}) = 5$. The other partitions which step 6 identifies are shown in Figure 7. Thus, R(2,v) is 5 and there are three vertex m-centers for this graph.

D. DISCUSSION

It is very difficult to make quantitative statements regarding the efficiency of this algorithm versus the "brute force" method. Too much depends on the characteristics of the graph under consideration. At worst, this algorithm might require one to calculate $\underline{R}_M$ for all possible partitions, just as the exhaustive approach does. At best, the algorithm can require as few as $(\underline{N}-1)!/(m-1)!(\underline{N}-m)!$ calculations of $\underline{R}_M$, (where $\underline{N}$ is the number of nodes in N). This represents $(m/\underline{N})$ of the $(\underline{N})!/m!(\underline{N}-m)!$ possible $\underline{R}_M$ calculations.

IV. THE ABSOLUTE M-CENTER PROBLEM

The ideas presented in this section were motivated by the author's realization that neither the absolute m-center algorithm of Reed [3], nor the algorithm due to Rosenthal and Smith [4] guaranteed an optimal solution. Although Rosenthal and Smith state that their iterative technique will arrive at the optimal solution, this is not true in general as will be shown with a simple example. Reed did not claim that his algorithm determined an optimal answer in all cases and was unable to provide a test to determine when it failed.

It is easily shown that the vertex radius of a graph is an upper bound on the absolute radius of the same graph [3]. That is,

$$R(v) \geq R(ab).$$

To the author's knowledge, however, a lower bound other than zero has never been derived for R(ab). The theorems which follow provide such a lower bound and delineate its usefulness in attacking the absolute m-center problem.

THEOREM 3:

Let $\underline{N}$ be the number of nodes in N. If $\underline{N} > 2$ for G(N,A), then the C(v,cp) must be an interior node of the central path of G(N,A). If $\underline{N} \leq 2$, the C(v,cp) must be an end node of the central path.

<u>Proof</u>: The case for $\underline{N} \leq 2$ is obvious. Suppose $\underline{N} > 2$ and the C(v,cp) is not an interior node of the central path. Let node q be the C(v,cp). Two possibilities occur.

Case 1. Assume node q is an end node of the central path. Let k be any interior node of the central path and j be the other end node of the central path. Let m be the end node of any other path in the central-path, spanning tree. The following inequalities must hold:

$$d(j,q)^* \geq d(m,q)^*,$$

$$d(j,q)^* > d(k,q)^*,$$

$$d(j,q)^* > d(m,k)^*.$$

The first inequality establishes the fact that

$$\max_{i \in N} d(i,q)^* = d(j,q)^*.$$

The second and third inequalities show that any central path interior node will have a smaller maximum distance to nodes of the central path spanning tree than q does. This contradicts the assumption that node q is the C(v,cp) since if node q actually was the C(v,cp) then

$$d(j,q)^* = \min_{f} \max_{i} d(i,f)^*,\ f,i \in N.$$

Case 2. Assume node q is either an end-node or an interior node on any path other than the central path. Let k be the first node interior to q which also is on the central path. Let m and n be the end nodes of the central path and j be the end node of any other path in the central-path, spanning tree. The following inequalities must hold:

$$\max \left[d(m,k)^*,\ d(n,k)^*\right] \geq d(k,j)^*,$$

$$d(k,m)^* < d(q,m)^*,$$

$$d(k,n)^* < d(q,n)^*.$$

This also contradicts the assumption that node q is the C(v,cp).

Since neither the assumptions of Case 1 nor those of Case 2 can be true, C(v,cp) must be an interior node of the central path.

THEOREM 4:

For a graph G(N,A),

$$R(ab,cp) \geq \tfrac{1}{2}R(v,cp).$$

<u>Proof</u>: Let vertices q and k be end nodes of the central path of G(N,A).

Case 1. If $\underline{N} \leq 2$,

$$R(ab,cp) = \tfrac{1}{2}R(v,cp).$$

Case 2. If $\underline{N} > 2$,

$$d(q,k)^* = d(q,C(v,cp))^* + d(C(v,cp),k)^*.$$

Assume that

$$d(q,C(v,cp))^* \geq d(C(v,cp),k)^*.$$

Now suppose that

$$R(ab,cp) \equiv \tfrac{1}{2}d(q,k)^* < \tfrac{1}{2}d(q,C(v,cp))^* \equiv \tfrac{1}{2}R(v,cp).$$

If this is true, then

$$d(q,k)^* < d(q,C(v,cp))^*,$$

which is impossible. Therefore,

$$R(ab,cp) \geq \tfrac{1}{2}R(v,cp)$$

must be true.

THEOREM 5:

For a graph G(N,A), let M be a partition, $M \subset N$. Let node j be a root of this partition. It must be true that

$$R_j(ab) \geq \tfrac{1}{2}R_j.$$

Proof: If $\underline{N}_j \leq 2$ the assertion is obviously true. Assume $\underline{N}_j > 2$. Let nodes k and q be the end nodes of the central path of $G(N_j, A)$. Define R_{j*} such that

$$R_{j*} = \max d(i,j)^*, \ i \in N_j.$$

Observe that

$$R_j \leq R_{j*}.$$

Case 1. Assume $j \equiv C_j(v,cp)$. Then

$$R_{j*} = \max [d(k,j)^*, d(j,q)^*] = R_j(v,cp).$$

Therefore, from Theorem 4,

$$R_j(ab) \geq \tfrac{1}{2} R_{j*}.$$

Case 2. Assume $j \equiv k$. Then

$$R_{j*} = d(j,q)^* = 2R_j(ab),$$

therefore

$$R_j(ab) = \tfrac{1}{2} R_{j*}.$$

Case 3. Assume j is an interior node of the central path of $G(N_j, A)$, but is not $C_j(v,cp)$. Then

$$R_{j*} = \max [d(k,j)^*, d(j,q)^*] < d(k,q)^*,$$

therefore

$$R_j(ab) > \tfrac{1}{2} R_{j*}.$$

Case 4. Assume j is any node on a path other than the central path of $G(N_j, A)$. Then

$$R_{j*} = \max [d(k,j)^*, d(j,q)^*] < d(k,q)^*.$$

Therefore

$$R_j(ab) > \tfrac{1}{2} R_{j*}.$$

The proof is complete since cases 1 through 4 are exhaustive.

This series of theorems is the foundation for the following absolute m-center algorithm. The algorithm first finds the minimum absolute partition radius among the vertex m-centers of the graph. The algorithm then uses the properties of the max-min node to find those partitions which could possibly improve the solution. If no such partitions exist the solution terminates.

A. THE ABSOLUTE M-CENTER ALGORITHM

For the graph G(N,A) with distance matrix $\|d_{ij}\|$:

1. Find all vertex m-centers of G(N,A) using the algorithm of section III.

2. Determine $\underline{R}_M(ab)$ for each vertex m-center, applying equation (3) and any absolute center algorithm which is convenient [3].

3. Set R(m,ab) equal to the minimum value found in step 2.

4. Determine the $\overline{V}(A,B/I)$ for G(N,A). For any $\overline{V}(A,B/I)$ such that

$$\overline{V}(A,B/I) \leq 2R(m,ab),$$

calculate $\underline{R}_M$ for any partition M such that

$$d(A,B) = \min_{v \in M} d(v,B).$$

5. Calculate $\underline{R}_M(ab)$ for any partition from step 4 such that

$$\underline{R}_M \leq 2R(m,ab).$$

If at any time during these calculations a value of $\underline{R}_M(ab)$ is found such that

$$\underline{R}_M(ab) < R(m,ab)$$

set $R(m,ab)$ equal to this new lower value. Do not calculate $\underline{R}_M(ab)$ for any M such that

$$\underline{R}_M > 2R(m,ab).$$

6. When all the appropriate $\underline{R}_M(ab)$ have been calculated the optimal solution is the extant $R(m,ab)$. Any partition having $\underline{R}_M(ab)$ equal to this value has a set of $C_j(ab)$'s which is an absolute m-center for this graph. Since any partition not already considered must have $\underline{R}_M(ab)$ greater than $R(m,ab)$ the algorithm terminates.

B. DISCUSSION AND EXAMPLE

Consider the graph and distance matrix used for the second example in section III. This is a very simple graph, ... deceptively simple.

Rosenthal and Smith claim that their iterative technique must converge to an optimal solution for the absolute m-centers of a graph. Their argument in support of this claim (on page 23 of [3]) hinges on the assumption an optimal solution has been found if it is not possible to improve the solution by moving some node from one subgraph to another. This is not true in all cases.

Consider the graph in Figure 5. When the absolute m-center algorithm of [4] was applied to this graph to find the absolute two-centers, subgraphs as shown in Figure 8 resulted with $R(2,ab) = 4$. The algorithm presented above arrives at a better value for $R(2,ab)$ for this graph.

It was found in the second example of section III that sets $\{2,5\}$, $\{1,4\}$ and $\{2,4\}$ qualify as vertex two-centers for the graph of Figure 5. Forming the appropriate associations it is found that $\{1,5\}$ produces the subgraphs depicted in Figure 8(a) while $\{2,5\}$ and $\{2,4\}$ produce the subgraphs of Figure 8(b), resulting in $R(m,ab) = 4$, as before.

From the distance matrix it is seen that nodes 1, 3, 4, and 5 have distances to $\overline{V}$ less than or equal to $2R(m,ab) = 8$. Thus $\underline{R}_M$ should be found for $\{4,5\}$, $\{3,5\}$, $\{2,5\}$, $\{1,5\}$, $\{4,3\}$, $\{4,2\}$, $\{4,1\}$, $\{1,2\}$, $\{1,3\}$ and $\{3,2\}$. $\underline{R}_M$ and $\underline{R}_M(ab)$ have already been calculated for three of these partitions so that they need not be calculated again. Figure 9 shows the calculation of $\underline{R}_M$.

Calculating $\underline{R}_M(ab)$ for $\{4,5\}$, the subgraphs of Figure 10 are produced, giving $\underline{R}_M(ab) = 3\frac{1}{2}$. This is a better solution and immediately removes $\{1,5\}$, $\{4,3\}$ and $\{3,2\}$ from consideration. Calculation of $\underline{R}_M(ab)$ for $\{3,5\}$, $\{1,2\}$ and $\{1,3\}$ gives the same result as $\{4,5\}$. The algorithm terminates with $R(m,ab) = 3\frac{1}{2}$.

The solution process appears much more difficult and involved than it actually is. However, once the purpose of each step is understood, hand calculations can be made rapidly.

The solution produced by the algorithm must be optimal. It starts with a solution minimally bounded from above by $R(m,v)$ and then tests all m-tuples which could possibly improve the solution, using a lower bound criterion based on Theorems 4 and 5 to select those m-tuples.

V. CONCLUSION

A. SUMMARY

This thesis develops a vertex m-center algorithm and an absolute m-center algorithm for $m > 1$. The vertex m-center algorithm is a search technique using the properties of the max-min node to guide the search. This algorithm should substantially reduce the work necessary to find the vertex m-centers of a graph, especially when it is not necessary to find all such m-centers, because it avoids calculating the vertex radius of all possible m-tuples partitioning the graph.

Several theorems are stated and proved which delineate a lower bound on a graph's absolute m-radius in terms of the vertex m-radius. This provides a test of optimality which, together with the demonstrated bounding properties of the max-min node, has been used to define a search technique for the absolute m-centers of a graph.

This search technique must find the optimal answer since it looks at any partition which can possibly improve the initially derived answer. The algorithm avoids unnecessary calculations by using the lower bound developed for the $\underline{R}_M(ab)$ to eliminate partitions which could not possibly give a better solution.

B. SUGGESTIONS FOR FUTURE RESEARCH

The author believes that a solution process for the absolute center of a graph can be developed which uses the properties of the

central-path end nodes to define an efficient algorithm. Such an algorithm has been developed by the author which successfully solves all examples given in [4]. It is assumed that the max-min node of a graph would always be an end-node of the central path, and proceeds to find the minimum, maximum length path through the graph having the max-min node as an end node. It has been shown that the process need not always work however, and more effort is necessary to establish the exact conditions under which it fails before attempting to use it generally.

It seems inefficient to generate a distance matrix for a graph and to use this matrix for further m-center calculations. It should be possible to develop algorithms similar to minimal spanning tree or shortest-route algorithms which work directly from the node-arc incidence relations and magnitudes to develop a partition having whatever properties are desired. The work of Reed [3] leads in this direction and could be extended profitably.

Finally, an effort should be made to extend the formulation context of the problem. Rosenthal and Smith [4] relate several possibilities and more should be available. One interesting idea would be to attribute probability densities to the arc magnitudes as is done in PERT and similar techniques.

APPENDIX A: FIGURES

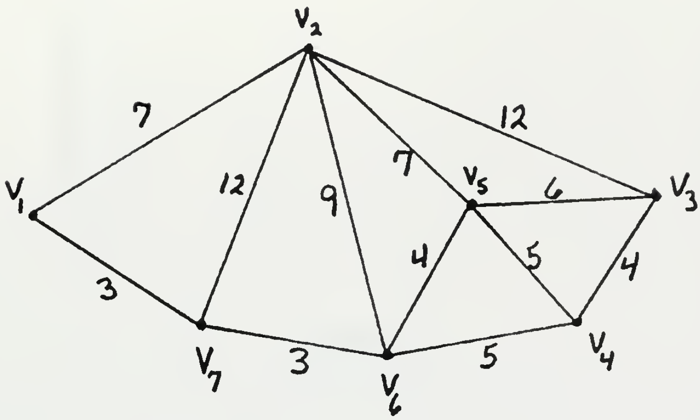


FIGURE 1

	1	2	3	4	5	6	7
1	0	7	15	11	10	6	3
2	7	0	12	12	7	9	10
3	15	12	0	4	6	9	12
4	11	12	4	0	5	5	8
5	10	7	6	5	0	4	7
6	6	9	9	5	4	0	3
7	3	10	12	8	7	3	0
min	3	$\underline{7}$	4	4	4	3	3

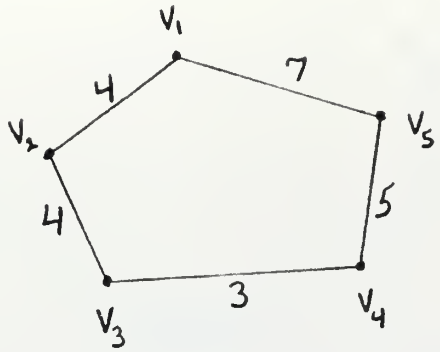
FIGURE 2

	1	2	3	4	5	6	7	$\underline{R}_M$
$\{1,2\}$	0	0	12	11	7	6	3	12
$\{2,3\}$	7	0	0	4	6	9	10	10
$\{2,4\}$	7	0	4	0	5	5	8	8
$\{2,5\}$	7	0	6	5	0	4	7	$\underline{7}$
$\{2,6\}$	6	0	9	5	4	0	3	9
$\{2,7\}$	3	0	12	8	7	3	0	12

FIGURE 3

	1	2	3	4	5	6	7	$\underline{R}_M$
$\{1,3\}$	0	7	0	4	6	6	3	$\underline{7}$
$\{1,4\}$	0	7	4	0	5	5	3	$\underline{7}$
$\{1,5\}$	0	7	6	5	0	4	3	$\underline{7}$
$\{1,6\}$	0	7	9	5	4	0	3	9
$\{1,7\}$	0	7	12	8	7	3	0	12
$\{3,5\}$	10	7	0	4	0	4	7	10
$\{4,5\}$	10	7	4	0	0	4	7	10
$\{5,6\}$	6	7	6	5	0	0	3	$\underline{7}$
$\{5,7\}$	3	7	6	5	0	3	0	$\underline{7}$

FIGURE 4



	1	2	3	4	5
1	0	4	8	11	7
2	4	0	4	7	11
3	8	4	0	3	8
4	11	7	3	0	5
5	7	11	8	5	0
min	4	4	3	3	<u>5</u>

FIGURE 5

	1	2	3	4	5	$\underline{R}_M$
$\{1,5\}$	0	4	8	5	0	8
$\{2,5\}$	4	0	4	5	0	<u>5</u>
$\{3,5\}$	7	4	0	3	0	7
$\{4,5\}$	7	7	3	0	0	7

FIGURE 6

	1	2	3	4	5	R_M
{1,4}	0	4	3	0	5	<u>5</u>
{2,4}	4	0	3	0	5	<u>5</u>
{3,4}	8	4	0	0	5	8

FIGURE 7

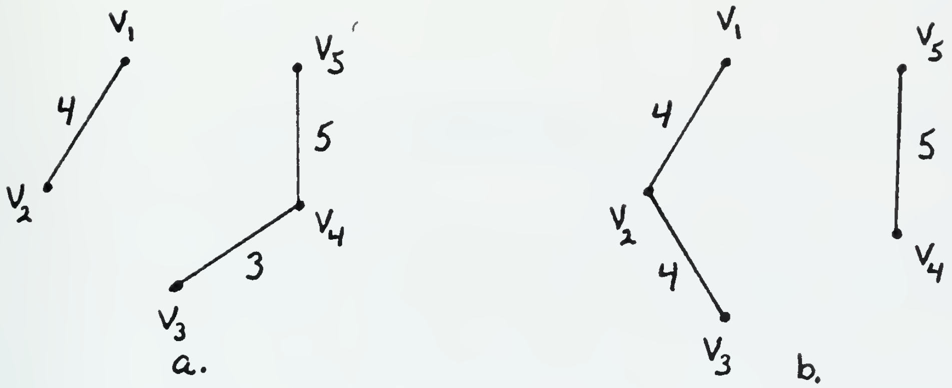


FIGURE 8

	1	2	3	4	5	R_M
{4,5}	7	7	3	0	0	7
{3,5}	7	4	0	3	0	7
{1,5}	0	4	8	5	0	8
{4,3}	8	4	0	0	5	8
{1,2}	0	0	4	7	7	7
{1,3}	0	4	0	3	7	7
{3,2}	4	0	0	3	8	8

FIGURE 9

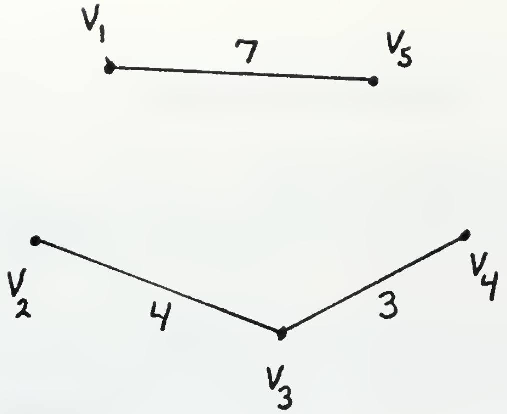


FIGURE 10

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4. Rosenthal, M.R., and Smith, S.B., The M-Center Problem, Paper No. TP2.1 presented at 31st National ORSA Meeting, New York, New York, May-June 1967.

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13. ABSTRACT

Solution algorithms are presented for the vertex m-center and the absolute m-center problem. Both algorithms use partitioning techniques. The algorithms use special properties of the max-min node to test for optimality. The vertex m-center algorithm establishes an order among all partitions of a graph according to the smallest vertex m-radius each partition can have. It then directs one to calculate the vertex m-radii only for those partitions which can provide a minimal vertex m-radius. The absolute m-center algorithm establishes an initial solution which may not be optimal. Other partitions are then tested against this solution to determine whether or not they provide a better solution. A point is reached at which no untested partition can improve the extant solution and the algorithm terminates.

KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
-center						
ertex m-center						
osolute m-center						
artition						

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